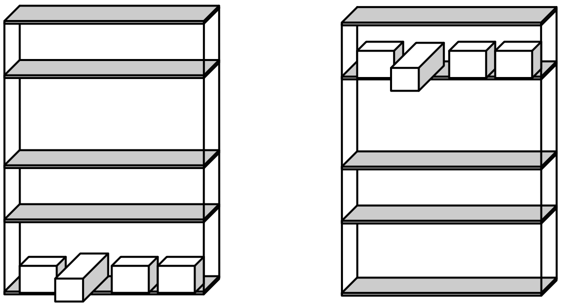
Centre of Mass

A central theme in mathematical modelling is that of reducing complex problems to simpler, and hopefully, equivalent problems for which mathematical analysis is possible. The concept of centre-of-mass is one such mathematical device for reducing the complexity of a problem to a more tractable system for which an understanding can be attempted in terms of mathematics.

The question of why loading a set of shelves from the top is more likely to induce an accident than if loaded first from the bottom is one such problem addressed by reducing the shelves and boxes to a single mass at a location through which the stability of the shelves can be understood.

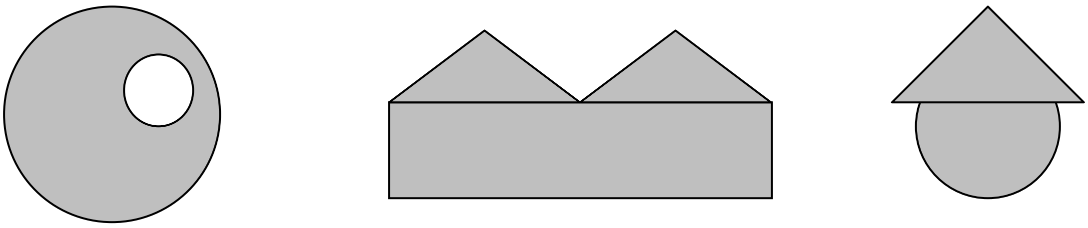


The centre-of-mass for a rigid body is central to almost all the solutions so far seen in this text. Each time a ladder is represented by a uniform rod, or a cricket ball is modelled as a particle, the essential idea behind centre-of-mass is deployed, namely, there exists a point in space through which the weight of these bodies acts. The physical dimensions are then only important in terms of the turning effect resulting from the rigid nature of the bodies.

While the centre of mass can be determined for 3D objects, the subject will be developed only for 1D and 2D rigid bodies. A 2D rigid body is referred to in mechanics as a lamina, which is characterised by a body with some mass and having an appreciable plane area, but negligible thickness. Discussions will be further limited to uniform lamina and networks of uniform rods.

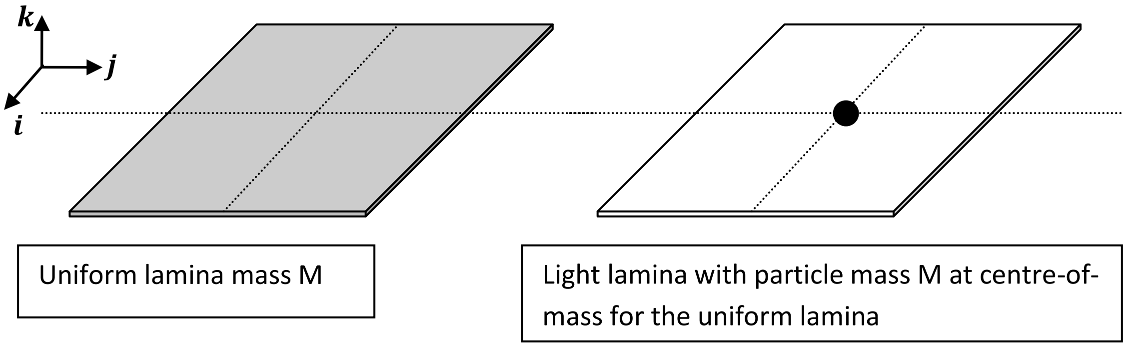
The significance of the term *uniform* is the centre of mass for a uniform rigid body can be determined based on the geometric lines of symmetry. Since the centre of mass for a uniform lamina must lie on a line of geometric symmetry, two or more geometric lines of symmetry cross at the position for the centre of mass. Thus, a uniform lamina with the geometry of a circle will have the centre of mass coinciding with the geometric centre for the circular shape.

Provided a complex lamina can be broken down into a set of shapes for which the centre of mass is known, the centre of mass for complex shaped lamina can be determined from the techniques described below.



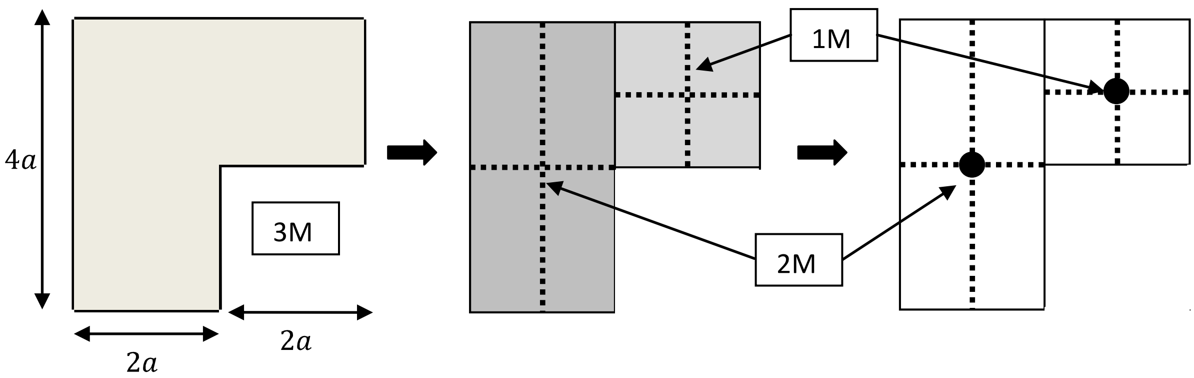
Calculating the centre of mass is performed by replacing the uniform lamina by a light lamina for which a single particle of mass equal to the mass of the uniform lamina is attached to the light lamina such that the turning effect under the influence of gravity for the two laminas about any line within the plane of these laminas is the same.

For a square uniform lamina lying in the horizontal plane, the turning effect of the lamina would be the same as that of a particle positioned at the point where two axes of symmetry cross, with the same mass as the uniform lamina.



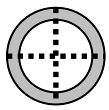
When discussing turning effects in 2D, the significant statement is the turning effect is now about a *line* rather than a *point*. For 1D problems, where an object is modelled using a rigid rod, the turning effect was considered to be about a point. Another way of thinking about the 1D concept of rotation about a point is that the point is the cross-section of a line perpendicular to the rod and the plane of the paper on which the rod is drawn. In this sense moments, even for 1D problems, were always about a line, not a point.

For more complicated shaped laminas, the problem of determining the centre of mass is that of reducing the lamina to shapes for which centre of masses are known, and then determining the centre of mass for a set of particles with masses and positions determined by these simpler shapes. For example, a square lamina with a quadrant missing could be broken down into a rectangle and a square. The centre of mass for the rectangle and square are determined by geometric symmetry, and these component lamina replaced by particles of equivalent masses.

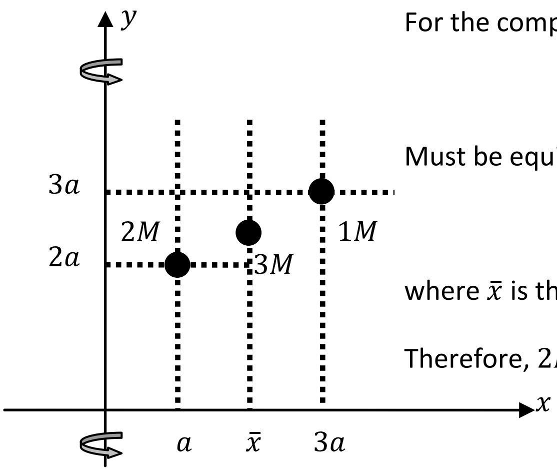


The centre of mass for the original shape is then determined from the centre of mass for a set of particles of different masses attached to a light lamina.

The light lamina to which these particles are attached is only conceptual and can be viewed as any shape we wish, which is important for situations where the centre of mass lies outside the boundary of the original lamina. The centre of mass for a hollow object, such as a washer, is not within the material part of the washer.



If a mechanical system can be reduced to a set of particles of known mass and spatial separation, then the centre of mass for the original rigid body can be calculated by summing the moments for each particle about any line we choose, and then calculating the distance from the line for a particle equal in mass to the total mass such that the moment of the particle of total mass is the same as the sum of the moments for all the particles. The sum of moments must account for a specified rotational sense with respect to the chosen line.



For the component particles, moments about the y-axis:

$$2Mg \times a + Mg \times 3a$$

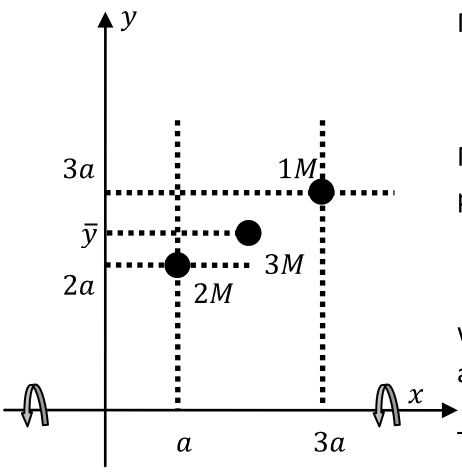
Must be equivalent to the moment for the total mass particle:

$$(2Mg + Mg) \times \bar{x}$$

where $\bar{x}$ is the distance of the centre of mass from the y-axis.

Therefore, $2Mg \times a + Mg \times 3a = (2Mg + Mg) \times \bar{x}$

$$\Rightarrow \bar{x} = \frac{2M \times a + 1M \times 3a}{2M + M} = \frac{5}{3}a$$



Moments about the x-axis for component particles:

$$2Mg \times 2a + Mg \times 3a$$

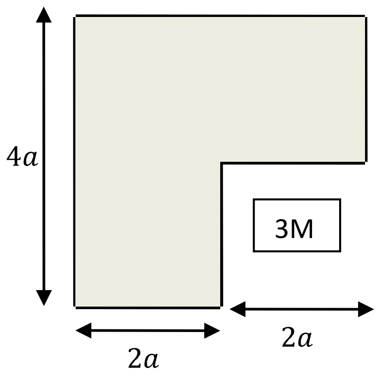
Must be equivalent to the moment for the total mass particle

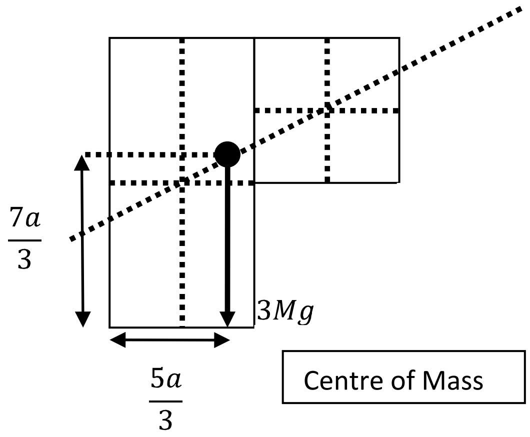
$$(2Mg + Mg) \times \bar{y}$$

where $\bar{y}$ is the distance of the centre of mass from the x-axis.

Therefore, $2Mg \times 2a + Mg \times 3a = (2Mg + Mg) \times \bar{y}$

$$\Rightarrow \bar{y} = \frac{2M \times 2a + M \times 3a}{2M + M} = \frac{7}{3}a$$





In general, provided a lamina can be reduced to a set of particles of known mass $\{m_1, m_2, m_3, \cdots, m_n\}$ and positions $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)\}$, applying the same logic for 2D shapes based on moments about the coordinate axes chosen for the shape, the centre of mass can be determined as a weighted average of position:

$$\bar{x} = \frac{m_1 \times x_1 + m_2 \times x_2 + m_3 \times x_3 + \cdots + m_n \times x_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

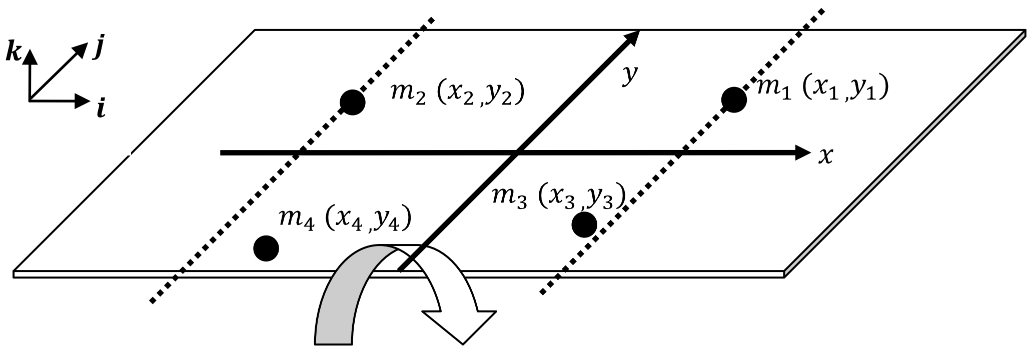
If the total mass $M = \sum_{i=1}^{n} m_i$,

$$\bar{x} = \frac{1}{M}\sum_{i=1}^{n} m_i \times x_i$$

Similarly,

$$\bar{y} = \frac{1}{M}\sum_{i=1}^{n} m_i \times y_i$$

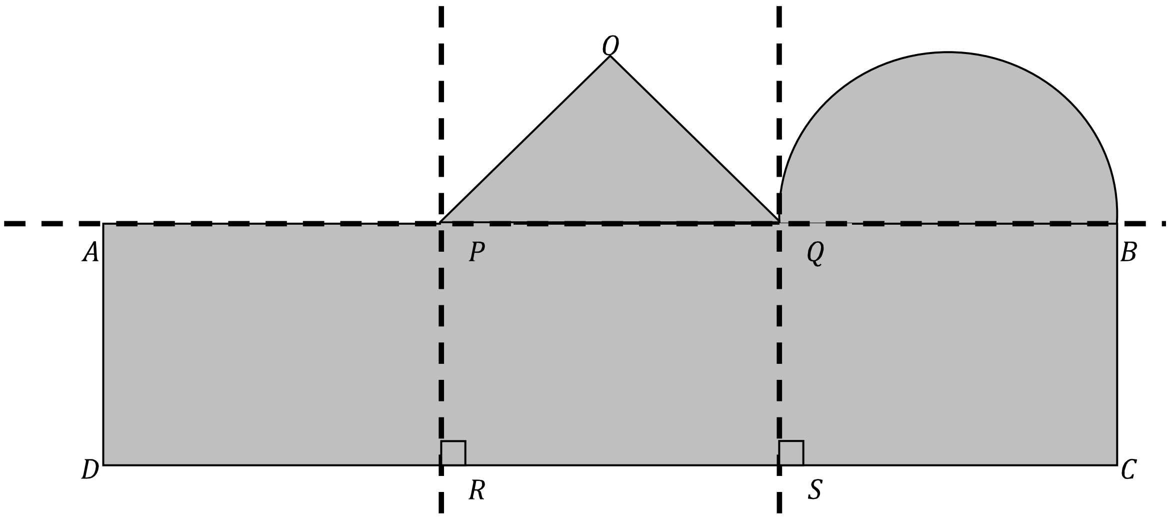
Note, depending on where the coordinate axes are chosen for a particular shape, the particle coordinates may be positive or negative. These positive and negative coordinate values correspond to the convention used for the rotational sense of the moments underlying these formulae.



Example

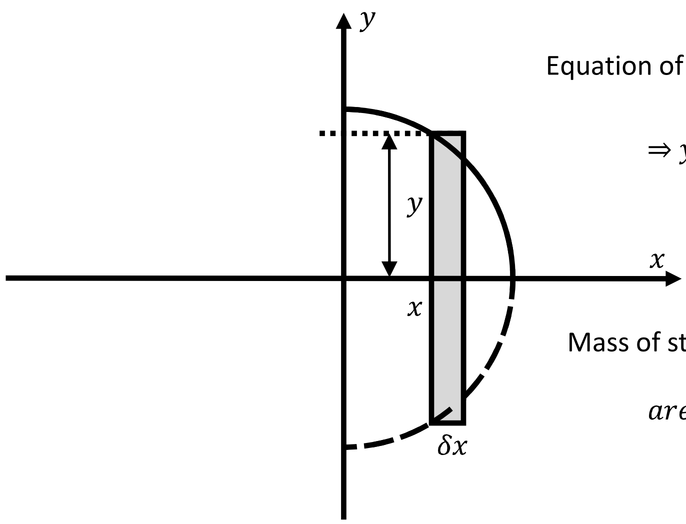
A uniform plane lamina constructed from a rectangle connected to a semi-circular section and an isosceles triangle POQ with dimensions $AB = 18a$, $PQ = QB = 6a$, $PO = QO = 3a\sqrt{2}$ and $BC = 4a$.

Show that the centre of mass for the lamina is $\frac{123\ \mathrm{a}}{75+4.5\pi}$ below the line AB.



The lamina is constructed from three components: the rectangle $ABCD$, a semi-circle and a triangle connected to the rectangle at PQ and QB. The first step is to calculate the centre of mass for a semi-circle using calculus. The triangle also needs further analysis, but the centre of mass for the rectangle is determined from geometric symmetry.

Centre of Mass for a Semi-circle



Equation of a circle $x^2 + y^2 = r^2$

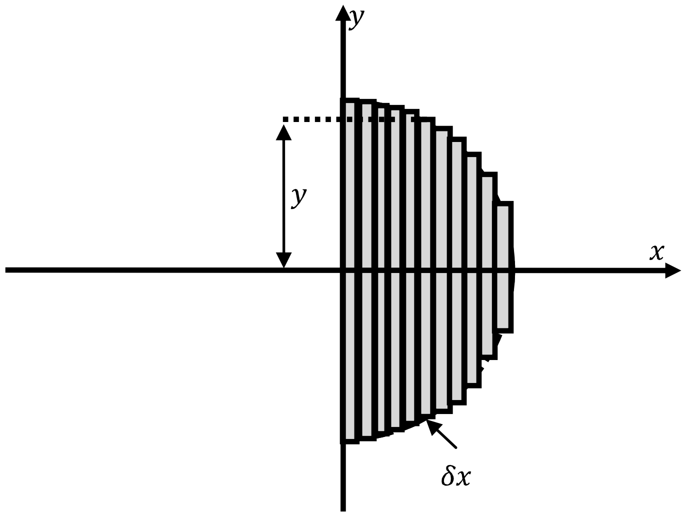
$$\Rightarrow y = \sqrt{r^2 - x^2}$$

Mass of strip with area density ρ is

$$area \times \rho = \delta x \times 2y \times \rho$$

By geometric symmetry, the centre of mass must lie on the x-axis for a semi-circle radius r obeying the functional form $y = \sqrt{r^2 - x^2}$ for $0 \leq x \leq r$.

Calculating distance for the centre of mass from the y-axis for a semi-circle of uniform density per unit area ρ is performed using the concepts of summing moments in exactly the same way moments are calculated for sets of particles. The semi-circle is approximated by a set of rectangular regions of width δx and height $2y$.



By symmetry, the centre of mass for these rectangles lies on the x-axis and the mass for each rectangle positioned x_i from the y-axis is

$$m_i = \delta x \times 2y_i \times \rho$$

Where $y_i = \sqrt{r^2 - x_i{}^2}$.

The total mass for the semi-circle is known exactly, and is $M = \frac{\pi r^2}{2}\rho$ acting through the centre of mass for the semi-circle at the, yet to be determined, position $\bar{x}$ from the y-axis.

If the centre of mass for the semi-circle is $\bar{x}$ from the y-axis, then an approximation to the centre of mass will be

$$\bar{x} \approx \frac{1}{M}\sum_{i=1}^{n} m_i \times (x_i + \delta x) = \frac{1}{M}\sum_{i=1}^{n} \delta x \times 2y_i \times \rho \times (x_i + \delta x)$$

$$\bar{x} \approx \frac{1}{\frac{\pi r^2}{2}\rho}\sum_{i=1}^{n} \delta x \times 2\sqrt{r^2 - x_i{}^2} \times \rho \times (x_i + \delta x) = \frac{2}{\pi r^2}\sum_{i=1}^{n}\left[2x_i\sqrt{r^2 - x_i{}^2}\delta x + 2\sqrt{r^2 - x_i{}^2}(\delta x)^2\right]$$

In the digital age, such an expression would be sufficient to obtain an answer within the precision achieved by a calculator or a computer, however, calculus can be used to derive an exact expression for the centre of mass.

If the number of rectangles used in this approximation is allowed to increase, the width δx for each rectangle decreases, thus as rectangles are added to the approximation, the area and therefore the mass for each rectangle gets smaller. By increasing the number of rectangles, the number of small masses becomes ever larger, and understanding the consequences of adding more and more of ever smaller items is precisely the stuff of calculus. Mechanics, differentiation and integration are strongly linked.

To obtain the centre of mass for the semi-circle, as $\delta x \to 0$ the summation tends to the integral (the sum involved in $(\delta x)^2$ must go to zero as $\delta x \to 0$)

$$\bar{x} = \frac{4}{\pi r^2}\int_0^r x\sqrt{r^2 - x^2}dx$$

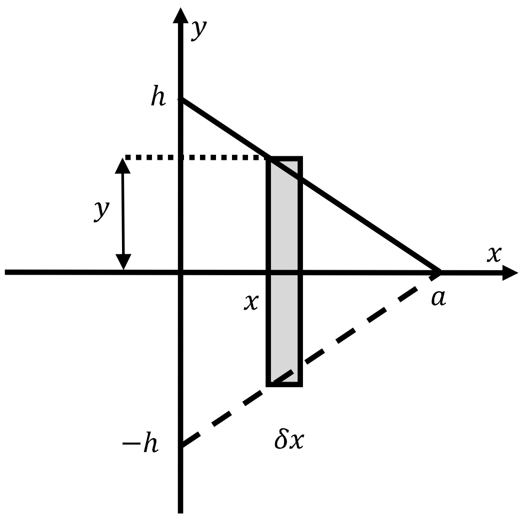
Using the substitution $u^2 = r^2 - x^2 \Rightarrow 2u\,du = -2x\,dx$, with limits from $u = r$ to $u = 0$.

$$\int_0^r \sqrt{r^2 - x^2}\,xdx = -\int_r^0 \sqrt{u^2}\,u\,du = \int_0^r u^2\,du = \left[\frac{u^3}{3}\right]_0^r = \frac{r^3}{3}$$

$$\therefore\ \bar{x} = \frac{4}{\pi r^2}\left[\frac{r^3}{3}\right] = \frac{4r}{3\pi}$$

This result is a standard result from a host of other standard centres of mass for uniform laminas, all of which will be provided under examination conditions, but nevertheless, illustrates where a desire to understand a physical object leads to the concepts of calculus and therefore should be seen as the motivation for studying techniques of integration in other mathematics courses.

Centre of Mass for an Isosceles Triangle



Equation of a line

$$\Rightarrow y = -\frac{h}{a}x + h$$

Mass of strip with area density ρ is

$$area \times \rho = \ \delta x \times 2y \times \rho$$

By symmetry, the centre of mass for an isosceles triangle lies on the x-axis and the mass for each rectangle positioned x_i from the y-axis is

$$m_i = \delta x \times 2y_i \times \rho$$

Where $y_i = -\frac{h}{a}x_i + h$.

The total mass for the isosceles triangle is $M = \frac{a \times 2h}{2}\rho = \rho ha$ acting through the centre of mass for the triangle at the position $\bar{x}$ from the y-axis.

If the centre of mass for the triangle is $\bar{x}$ from the y-axis, then an approximation to the centre of mass will be

$$\bar{x} \approx \frac{1}{M}\sum_{i=1}^{n} m_i \times (x_i + \delta x) = \frac{1}{M}\sum_{i=1}^{n} \delta x \times 2y_i \times \rho \times (x_i + \delta x)$$

$$\bar{x} \approx \frac{1}{ah\rho}\sum_{i=1}^{n} \delta x \times 2(-\frac{h}{a}x_i + h) \times \rho \times (x_i + \delta x)$$

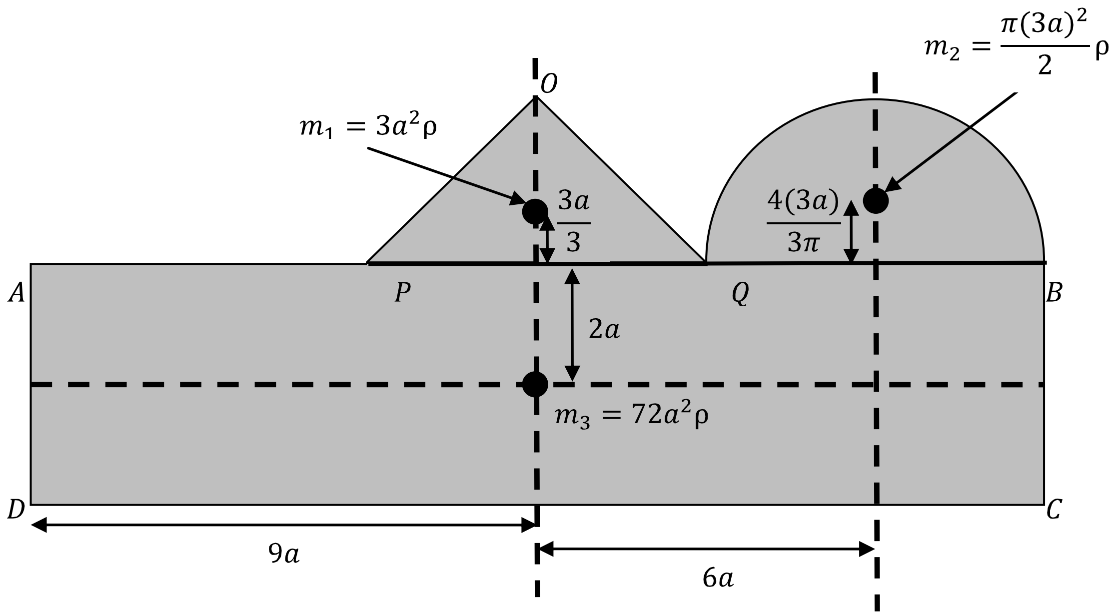
$$= \frac{1}{ah}\sum_{i=1}^{n}\left[2x_i(-\frac{h}{a}x_i + h)\delta x + 2(-\frac{h}{a}x_i + h)(\delta x)^2\right]$$

Again moving to the limit as $\delta x \to 0$

$$\bar{x} = \frac{2}{ha}\int_0^a x\left(-\frac{h}{a}x + h\right)dx$$

$$\Rightarrow \bar{x} = \frac{2}{ha}\int_0^a -\frac{h}{a}x^2 + hx\,dx = \frac{2}{ha}\left[-\frac{h}{3a}x^3 + \frac{h}{2}x^2\right]_0^a = \frac{2}{ha}\left[-\frac{h}{3a}a^3 + \frac{h}{2}a^2\right] = \frac{a}{3}$$

The centre of mass for the complex shape can now be reduced to a set of three particles located at the centres of mass for each of the component parts.



To calculate the distance of the centre of mass for the entire lamina from the line AB, the moment of the weights for these three particles about AB must be the same as the moment about AB of a single particle of total mass $M = 3a^2\rho + \frac{\pi(3a)^2}{2}\rho + 72a^2\rho$. Thus,

$$\bar{x} \times M = m_1 \times \mathrm{a} + m_2 \times \frac{4a}{\pi} + m_3 \times (-2\mathrm{a})$$

$$\bar{x} \times \left(3a^2\rho + \frac{\pi(3a)^2}{2}\rho + 72a^2\rho\right) = 3a^2\rho \times \mathrm{a} + \frac{\pi(3a)^2}{2}\rho \times \frac{4a}{\pi} + 72a^2\rho \times (-2\mathrm{a})$$

Since the centre of mass for the particle corresponding to the rectangle is on the opposite side of the line AB to the two particles corresponding to the triangle and the semi-circle, the moment for the rectangle is negative relative to the two particles above the line AB.

$$\bar{x} = \frac{a^3\rho(3 + 18 - 144)}{a^2\rho\left(3 + \frac{9\pi}{2} + 72\right)} = \frac{-123\ \mathrm{a}}{75 + 4.5\pi} = -1.38\ \mathrm{a}$$

The value for $\bar{x}$ is negative, therefore the centre of mass is below the line AB.

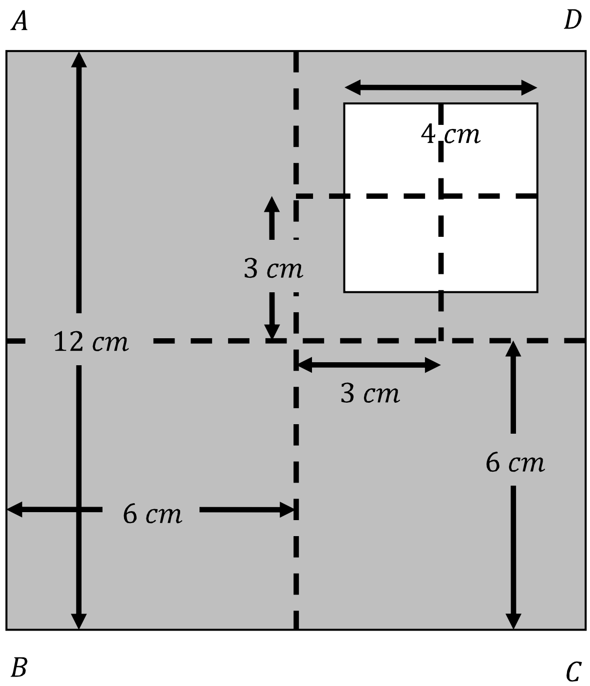
Example

A metal bracket is designed as a square ABCD of size 12 cm with a square of size 4 cm removed from one quadrant as shown in the diagram.

a) Find the distance of the centre of mass for the bracket from the side BC.

The bracket is suspended from A and hangs at rest.

b) Find the size of the angle between AB and the vertical.



The problem differs slightly from the previous example by virtue of a missing square of metal rather than shapes being added to a square. The problem could be approached by dividing the metal bracket into many smaller rectangles from which the centre of mass for the bracket could be calculated. However, by applying mathematical reasoning, the problem can be solved by considering three masses corresponding to:

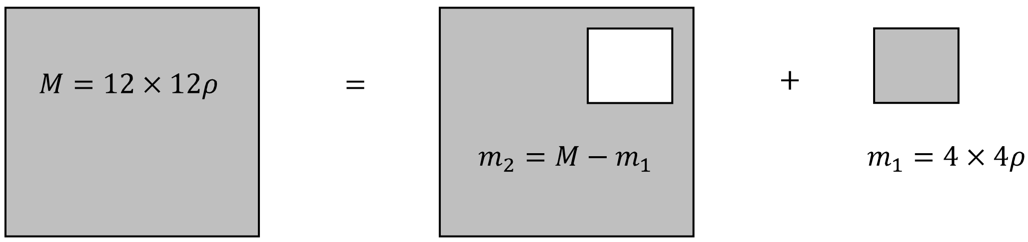
1. The mass of a square bracket without the removal of the smaller square.
2. The mass of the square corresponding to the missing smaller square.
3. The mass of the bracket with the smaller square removed.

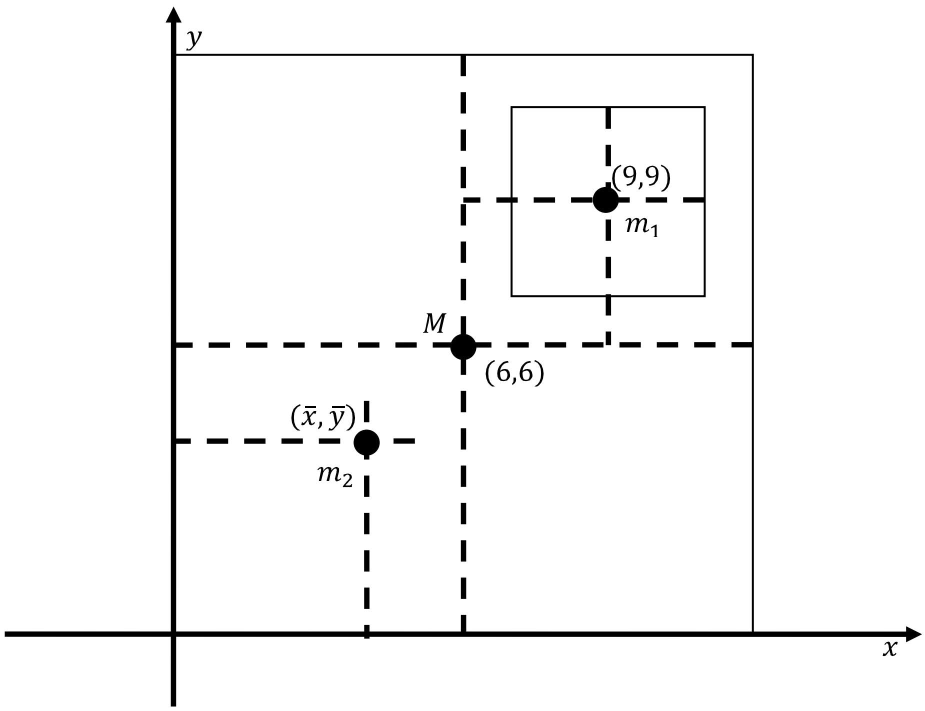
The reasoning is as follows. The bracket before the smaller square is punched out consists of the union of the bracket after the smaller square is removed and the small square itself. If the metal plate from which the bracket is made is considered to be constructed from the bracket plus the smaller square, the centre of mass for the square metal plate can be expressed in terms of the two component parts. The only difference is the unknown is now one of the component parts.

Since the centre of mass for the larger square is known and the centre of mass for the smaller square is also known, both obtained using the lines of symmetry for a square, the unknown centre of mass for the bracket can be determined.

a) Find the distance of the centre of mass for the bracket from the side BC.

Assuming the metal bracket can be modelled as a uniform lamina of density per unit area ρ, the masses for a square of size 12 cm by 12 cm, the smaller square of size 4 cm by 4 cm and the bracket are calculated as follows.





The same logic therefore applies as before, namely, the moments about the x-axis for the full square must be the same as the moments about the x-axis for the two component parts.

$$6 \times M = \bar{y} \times m_2 + 9 \times m_1$$

$$\Rightarrow \bar{y} = \frac{6 \times M - 9 \times m_1}{m_2}$$

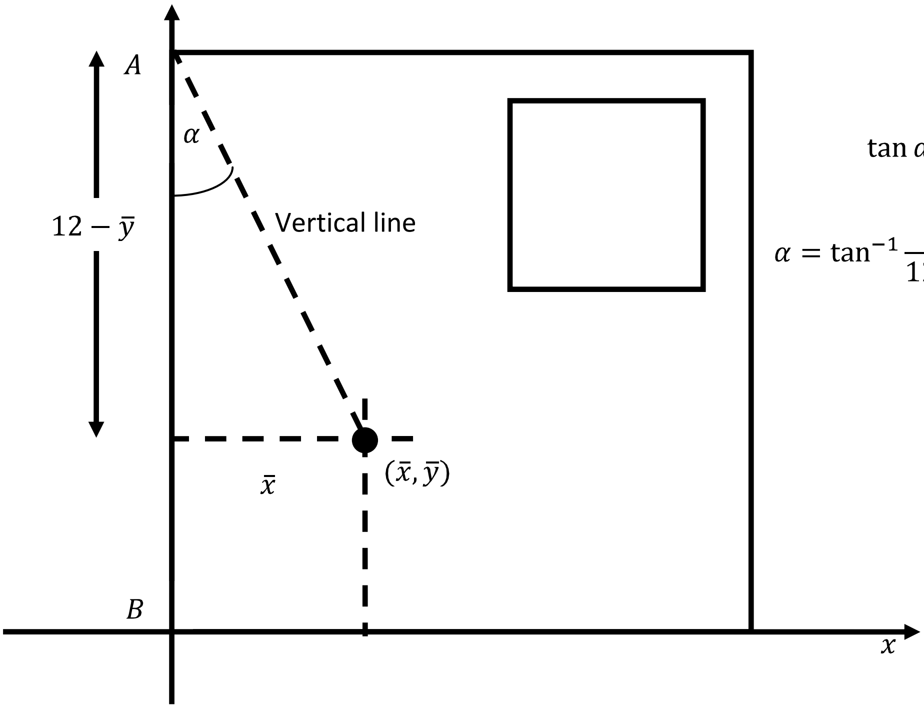
$$\Rightarrow \bar{y} = \frac{6 \times 144 - 9 \times 16}{128} = \frac{45}{8} = 5.625$$

By symmetry, $\bar{x} = \bar{y}$.

b) Find the size of the angle between AB and the vertical.

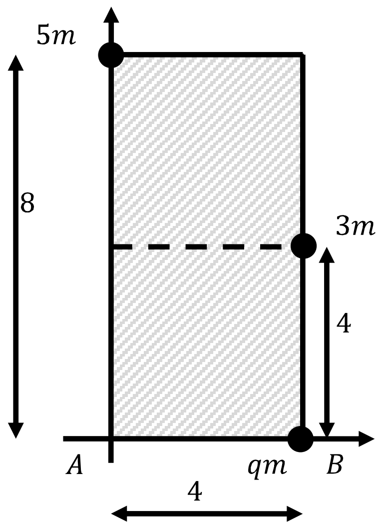
The lamina is suspended from the corner A and is assumed to be at rest. The practical method for determining the centre of mass for a metal bracket is to suspend the bracket from two different corners and each time mark the line on the bracket through the corner from which the bracket is suspended and a plumb line hanging vertically downwards. Two such lines marked on the metal bracket would intersect at the centre of mass.

If the bracket is suspended at rest from the corner A, then the mathematical solution is therefore obtained by drawing a line through A which passes through the centre of mass previously calculated.



$$\tan\alpha = \frac{\bar{x}}{12-\bar{y}}$$

$$\alpha = \tan^{-1}\frac{5.625}{12-5.625} = 41.4°$$



Example

Three particles are attached to a uniform rectangular lamina. The coordinates for the particles are $(0,8)$, $(4,4)$ and $(4,0)$ corresponding to masses $5m$, $3m$ and qm, respectively.

a) Given that the centre of mass for these three particles in the absence of the uniform lamina has x-coordinate equal to 2.75, calculate the value of q.
b) If the mass of the uniform lamina is $12m$, find the coordinates for the centre of mass of the combined system, namely, lamina and three attached masses.
c) If the combined system is freely suspended from the corner A , calculate the angle between AB and the horizontal.

Solution

a) Given that the centre of mass for these three particles in the absence of the uniform lamina has x-coordinate equal to 2.75, calculate the value of q.

By taking moments about the y-axis, the particle of mass $5m$ will be eliminated from the calculation while the particle specified as qm will be part of the equation. Thus, taking moments about the y-axis for the three particles in the absence of the lamina,

$$5m\,(0) + 3m\,(4) + \; qm\,(4) = (5m + 3m + qm)(2.75)$$

$$\Rightarrow \quad q(4 - 2.75) = 8(2.75) - 3(4)$$

$$\Rightarrow \quad q = \frac{10}{1.25} = 8$$

b) If the mass of the uniform lamina is $12m$, find the coordinates for the centre of mass of the combined system of lamina and three masses.

Since the first part of the question gave the x-coordinate for the centre of mass for the three particles attached to a light lamina, calculating the corresponding y-coordinate for the centre of mass offers a means of reducing the problem from a four mass to a two mass problem. The three masses attached to the lamina will be reduced to a single particle of mass $16m = 5m + 3m + 8m$ positioned at the calculated centre of mass for these three individual masses.

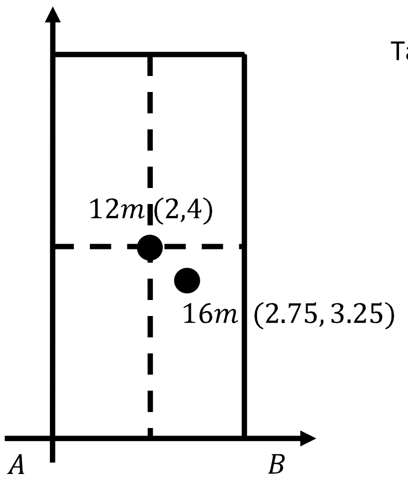
The y-coordinate for the three particles of mass $5m$, $3m$ and $8m$ is obtained by taking moments about the x-axis.

$$16m\ (\bar{y}) = 5m(8) + 3m(4) + 8m(0)$$

$$\bar{y} = \frac{40 + 12}{16} = \frac{52}{16} = 3.25$$

The centre of mass for the three particles is therefore $(2.75, 3.25)$. A particle of mass 16m positioned at $(2.75, 3.25)$ has the same moment as the three separate particles.

The centre of mass for the combined system of uniform lamina mass $12m$ and the three particles is obtained by finding the centre of mass for the two particle of mass $12m$ positioned at $(2,4)$ and $16m$ positioned at $(2.75, 3.25)$.



Taking moments about the y-axis:

$$28m\bar{x} = 12m(2) + 16m(2.75)$$

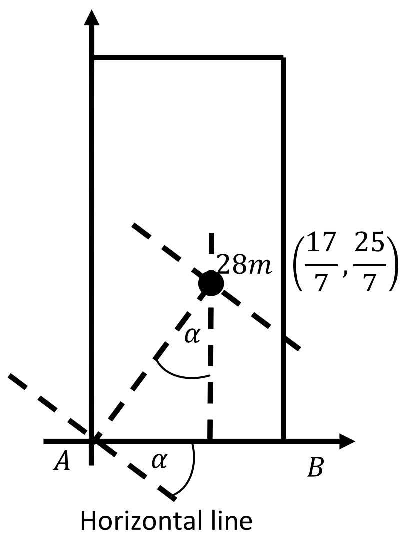
$$\bar{x} = \frac{24 + 44}{28} = \frac{17}{7}$$

Taking moments about the x-axis:

$$28m\bar{y} = 12m(4) + 16m(3.25)$$

$$\bar{y} = \frac{48 + 52}{28} = \frac{25}{7}$$

c) If the combined system is freely suspended from the corner A , calculate the angle between AB and the horizontal.



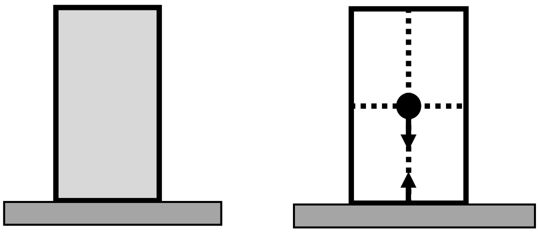
Suspending the combined system from A means the centre of mass will be vertically below A, therefore the angle between AB and the horizontal is the angle α marked on the diagram.

$$\tan\alpha = \frac{\frac{17}{7}}{\frac{25}{7}} = \frac{17}{25}$$

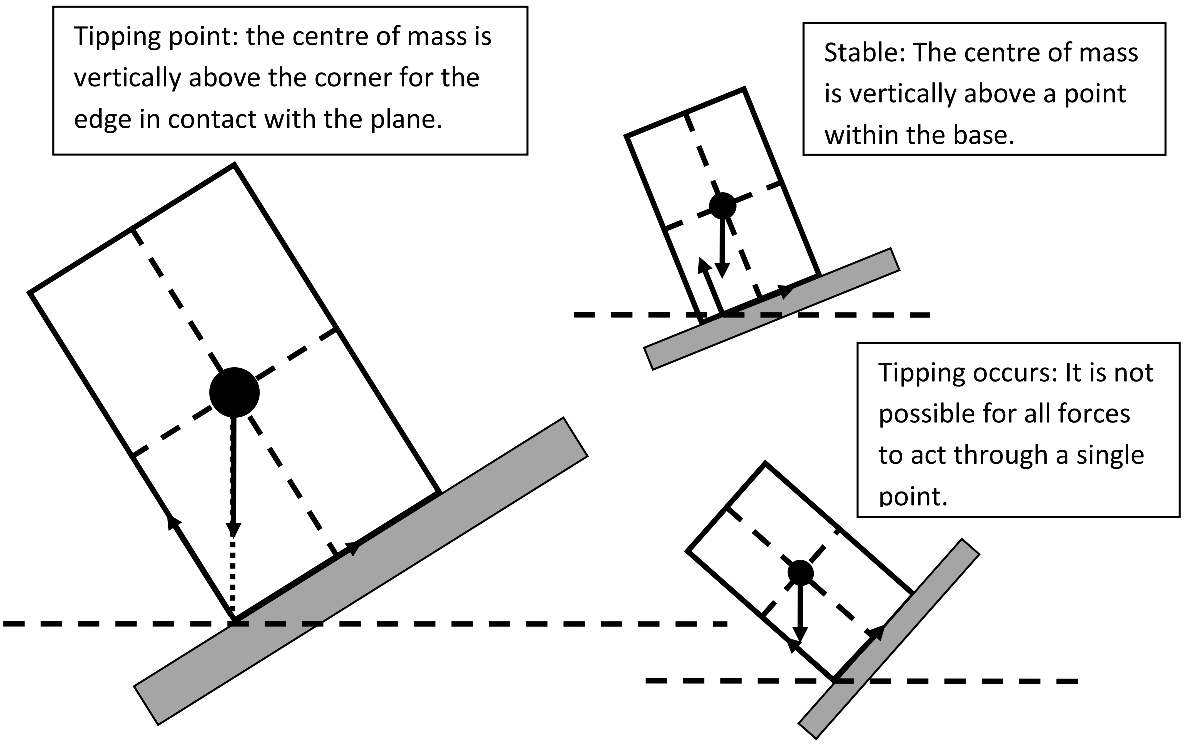
$$\alpha = 34.2°$$

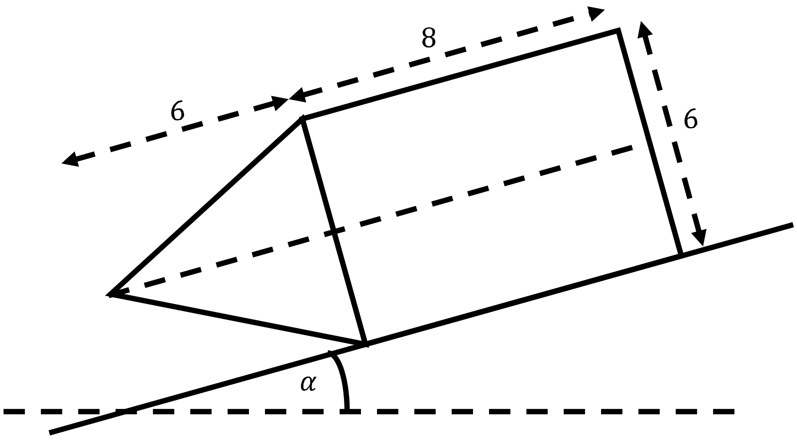
Toppling Points for a Lamina

If the reaction force to a lamina placed on a plane is modelled as a force acting perpendicular to the plane through the point vertically below the centre of mass for the lamina, the problem of stability with respect to toppling can be understood in terms of turning effects about the point vertically below the centre of mass.



Placing a lamina on an inclined plane causes the position for the reaction force to change. For a rough surface a friction force must also act through the point vertically below the centre of mass. If all the forces acting on a body act through the same point, since the distance from that point to the line of action for all the forces is zero, the moment about that point is zero. For a lamina in contact with a plane, the reaction and friction forces can only act through a point of the lamina in contact with the plane, thus provided the centre of mass is vertically above any point in contact with the plane, the sum of the moments for all the forces must be zero and therefore the body modelled by the lamina will be stable with respect to toppling.





Example

A uniform lamina is constructed from an isosceles triangle with base 6 cm and height 6 cm connected to a rectangle of size 8 cm by 6 cm as shown in the diagram.

a) Determine the distance of the centre of mass from the common edge between the triangle and the rectangle.

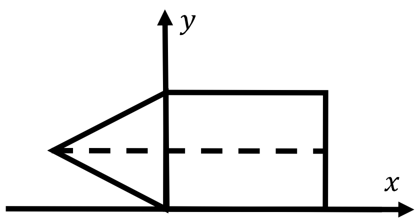
The lamina is placed on an inclined plane which makes an angle α to the horizontal.

b) Determine the largest angle α for an inclined plane such that the lamina does not topple.

The solution for part a) is required in the solution of part b). The angle at which the lamina is at the point of tipping occurs when the centre of mass is vertically above the point on the base of the lamina in contact with the plane about which the lamina will rotate when toppling.

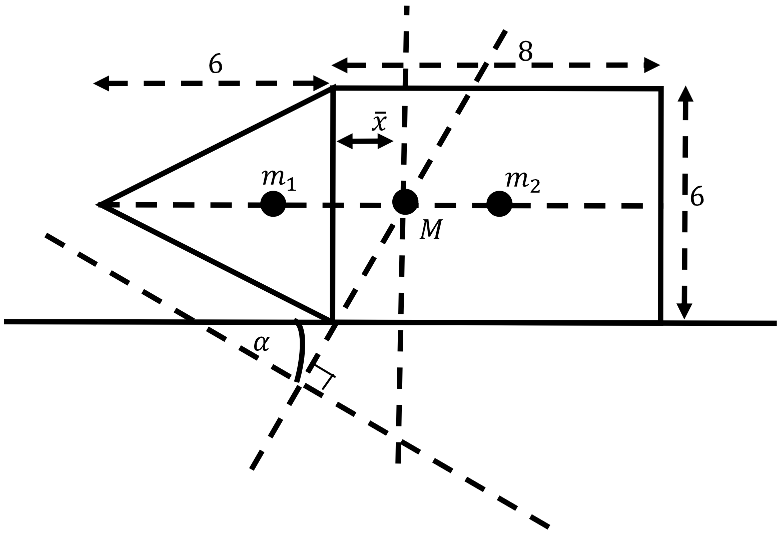
a) Determine the distance of the centre of mass from the common edge between the triangle and the rectangle.

Let the distance of the centre of mass for the lamina from the line joining the triangle to the rectangle be $\bar{x}$. Since the lamina has a line of symmetry passing through the mid-point of the triangle base and the mid-point of the rectangle edge parallel to the triangle base, the centre of mass for a uniform lamina must lie on this line of symmetry. The coordinate axes can be positioned with respect to the lamina as follows.



The y coordinate of centre-of-mass for the triangle, the rectangle and the combined lamina all have the same value, namely, 3.

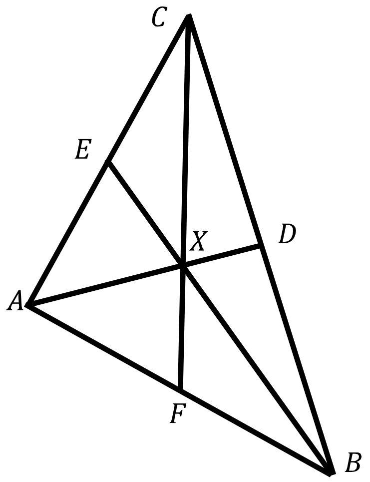
The distance $\bar{x}$ is then calculated to be the same as the x coordinate for the centre of mass for the lamina.



The centre of mass for the lamina is calculated using prior knowledge about the centre of mass of a triangular uniform lamina and the centre of mass for a rectangular uniform lamina. If the mass per unit area is ρ, then the information used in calculating the centre of mass for the combined lamina is as follows.

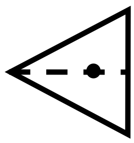
$$mass = area \times \rho$$

The centre of mass for any triangle, not just an isosceles triangle, lies on the median line passing through a vertex and the mid-point of the opposite side to the triangle. It can be shown that the distance from the mid-point of a side to the centre of mass is a third the length of the median line passing through the opposite vertex. The distance from the y-axis for the triangular lamina of height 6 cm is therefore $\frac{6}{3} = 2$ to the left of the y-axis so the x-coordinate for the centre of mass is -2.



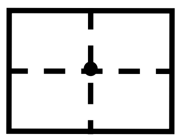
Median lines drawn from the mid-point of a side to the opposite vertex intersect at the centre of mass for a scalene triangle.

$$XD = \frac{1}{3}AD$$



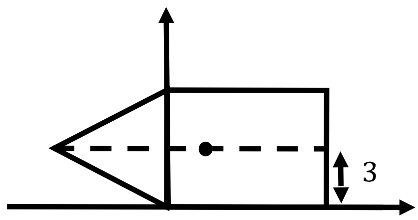
$$m_1 = \frac{6 \times 6}{2} \times \rho = 18\rho$$

$$C\ of\ Mass\ (-2, 3)$$



$$m_2 = 8 \times 6 \times \rho = 48\rho$$

$$C\ of\ Mass\ (4, 3)$$



$$M = m_1 + m_2 = 66\rho$$

$$C\ of\ Mass\ (\bar{x}, 3)$$

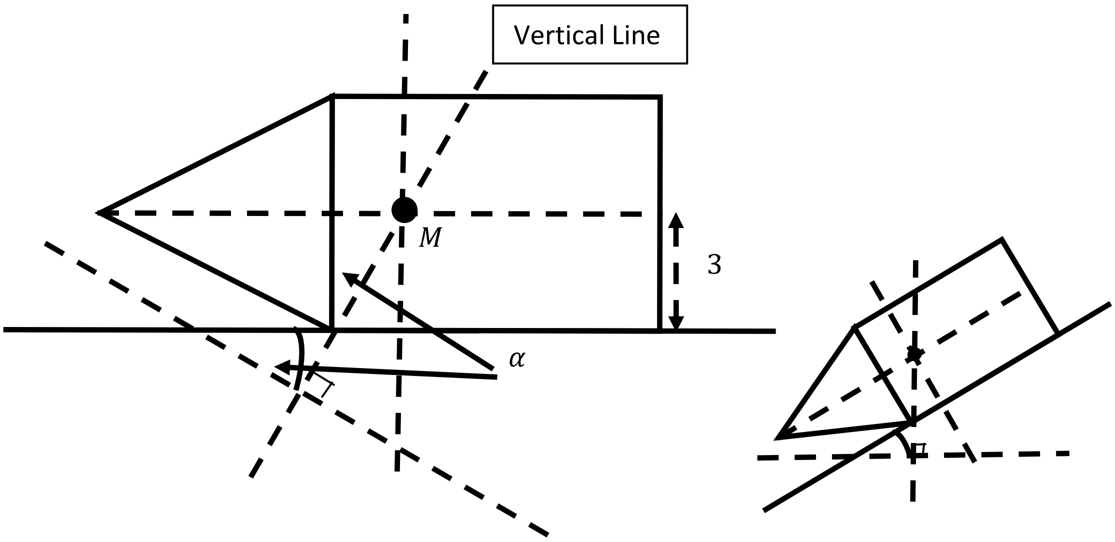
Taking moments about the y-axis,

$$M \times \bar{x} = m_1 \times (-2) + m_2 \times (4)$$

$$\bar{x} = \frac{18\rho \times (-2) + 48\rho \times (4)}{66\rho} = \frac{156}{66} = \frac{26}{11}\ cm$$

b) Determine the largest angle α for an inclined plane such that the lamina does not topple.

The tipping point for the lamina placed on an inclined plane is determined by assuming the centre of mass is vertically above the corner about which rotation can occur.



The angle for the inclined plane is therefore given by

$$\tan\alpha = \frac{\bar{x}}{\bar{y}} = \frac{\frac{26}{11}}{3} = \frac{26}{33}$$

$$\alpha = \tan^{-1}\left(\frac{26}{33}\right) = 38.2°$$

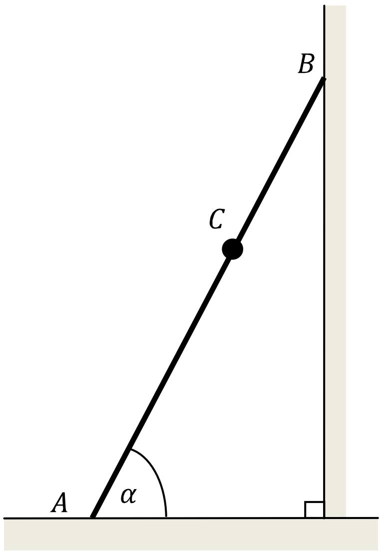
Solving Statics Problems

The types of problem associated with statics include mechanical systems which do not move. For example a ladder placed against a wall, where it is important for anyone climbing the ladder that it does not move. These problems are analysed in terms of forces, friction and moments of forces. Techniques used to analyse a statics problem in terms of forces are also used as part of dynamics when determining the resultant force acting on a particle in motion. Statics at the level discussed in this text is concerned with rigid bodies, not just particles, and as such requires the consideration of turning effects of forces acting on rigid bodies. Turning effects of forces on rigid bodies are dealt with using moments of forces which include an understanding of where the centre of mass for a rigid body is located, which in turn is determined using moments of forces. The problems included in this section therefore are not only the traditional statics problems, but will be supplemented by elementary determination for the centre of mass for a mechanical system.

Example

A painter places a ladder AB of length $5\ m$ and mass $20\ kg$ against a smooth vertical wall and on a rough horizontal floor, such that the ladder is in a vertical perpendicular plane with respect to the wall. The painter of mass $80\ kg$ stands at point C on the ladder such that $BC = 2\ m$. The coefficient of friction between the ground and the ladder is $\frac{9}{20}$. If the ladder is on the point of slipping, by modelling the ladder as a uniform rod and the painter as a particle,

a) Show that the magnitude of the frictional force of the ground on the ladder is $441\ N$.
b) Determine the angle α made by the ladder with the horizontal floor.

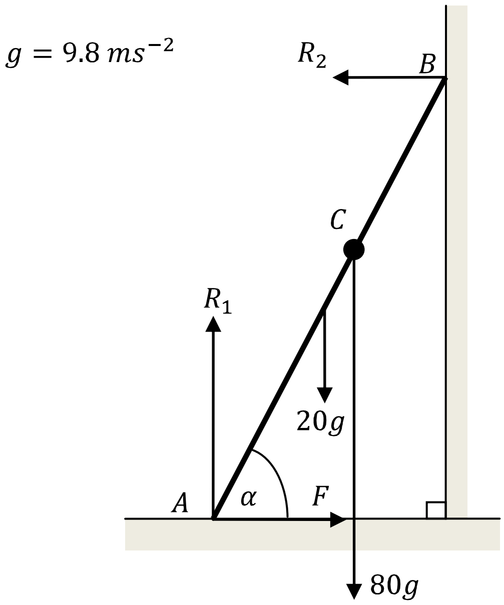


The problem as stated models the ladder as a rigid uniform rod of mass $20\ kg$ and the painter as a particle of mass $80\ kg$ located at C. The concept of a particle allows a complex shape like that of a painter to be reduced to a point mass equivalent to the mass of the painter. Clearly a painter is not rigid, thus replacing the painter by a point is a relatively crude form of mathematical modelling. The rigid uniform rod similarly allows the weight of the ladder to act through the geometric centre of the rod, which is, by virtue of the uniform rod assumption, the centre of mass for the rod. A ladder replaced by a rod is a reasonably good mathematical model, and one that is more realistic than a painter approximated by a particle.

The term geometric centre is used to describe a point at which all lines of symmetry meet.

When solving problems involving ladders, the first step is to annotate a diagram with the forces acting on the mechanical system consisting of the ladder, the painter, the floor and the wall. Since the ladder is modelled as a uniform rod of length $AB = 5\ m$, the centre of mass for the ladder is

2.5 m from A. The particle is located 2 m from B, therefore is 3 m from A. The weight of the ladder is $20g\ N$ and the weight of the particle is $80g\ N$. In addition to these weights, since the ladder stands on a rough floor, the diagram must include a normal reaction force R_1 and a friction force F. The direction for the friction force should be to oppose any potential movement the ladder would make should the ladder begin to slip. The wall is smooth and therefore the contact point of the ladder with the wall has no friction force parallel to the wall surface, but must include a normal reaction force R_2.



If the ladder and the painter are at rest, then the resultant force in any direction must be zero and the sum of the moments about any point must be zero too.

a) Find the magnitude of the frictional force of the ground on the ladder.

The question states the ladder and painter are at the point where the ladder is about to slip, which in mechanics terms means the friction force is at the maximum possible value. Since friction is a passive force, the size of the friction force depends on the forces attempting to move the ladder, so in general, if μ is the coefficient of friction and R is the magnitude of the normal reaction force, then the friction force obeys the relationship

$$F \leq \mu R$$

And only attains the maximum value when in limiting equilibrium, that is, at the point just before the ladder slips. The maximum friction force is obtained using $F = \mu R$. Since the coefficient of friction between the ladder and the floor is given, namely, $\mu = \frac{9}{20}$, the solution for the magnitude of the friction force requires the determination of the normal reaction force R_1 at A.

For a body at rest, the resultant force in any direction must be zero, and in particular, since the normal reaction force at A acts vertically, resolving the forces vertically yields

$$R_1 - 20g - 80g = 0$$

$$R_1 = 20g + 80g = 100g$$

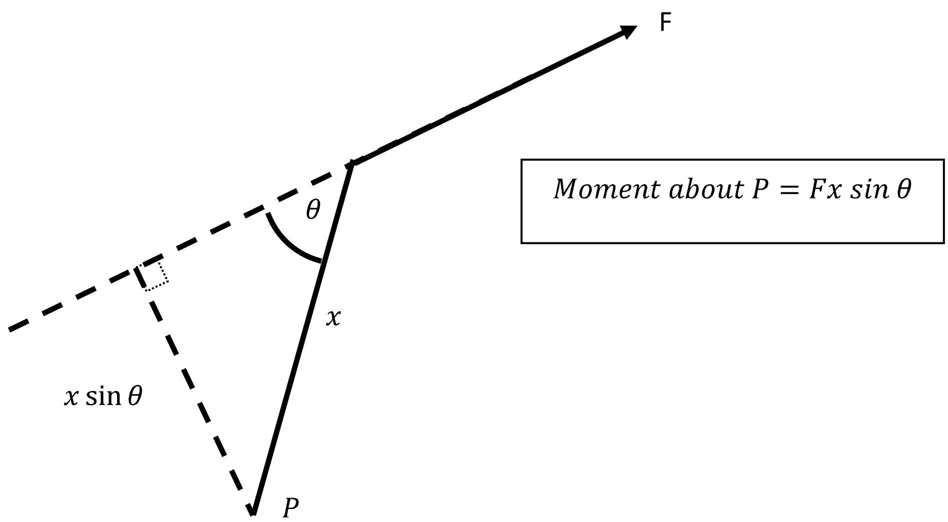
Thus, the magnitude of the friction force F is given by

$$F = \frac{9}{20}100g = 441\ N$$

b) Determine the angle α made by the ladder with the horizontal floor.

The angle α is obtained by generating equations involving lengths and forces. For a rigid body to be in equilibrium, the moments of the forces taken about any point must sum to zero. Since the condition for equilibrium must be satisfied for any point, the best point to choose is a point on the ladder which simplifies the resulting equations. Taking moments about A eliminates the normal reaction force and the friction force from the equations.

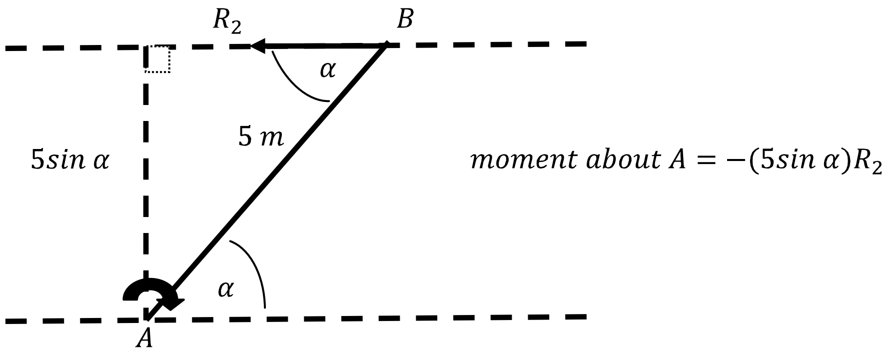
The moment of a force Fabout a point is the product of the magnitude of the force and the perpendicular distance from the point to the line of action of the force.



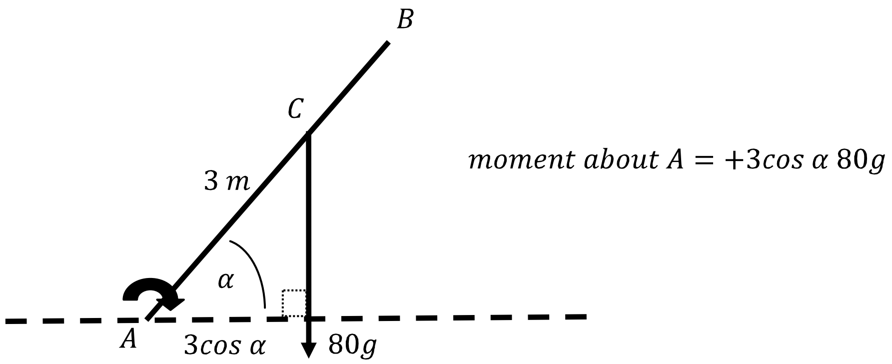
For the ladder, the distance from the point A to the line of action of the various forces will introduce the angle α. Further, since the line of action for both the friction and the normal reaction force at A both pass through A, the distance between A and these two forces is zero, hence the earlier statement asserting the friction and normal reaction forces at A are eliminated by choosing A as the point about which moments are taken.

Moments about A, taking clockwise direction as positive:

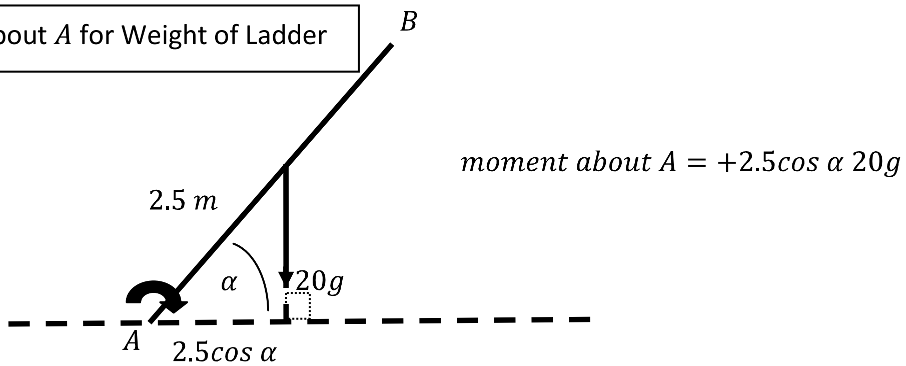
Moment about A for Normal Reaction Force at Wall



Moment about A for Weight of particle at C



Moment about A for Weight of Ladder



Therefore, summing the moments about A yields:

$$2.5cos\ \alpha\ 20g + 3cos\ \alpha\ 80g - (5sin\ \alpha)R_2 = 0$$

This equation involves two unknowns, therefore another equation is required involving R_2. Resolving the forces in the horizontal direction $F - R_2 = 0\ \therefore\ R_2 = F = 45g = 441\ N$.

$$\Rightarrow ((2.5)20 + (3)80) \times g \times cos\ \alpha = (5sin\ \alpha)(45 \times g)$$

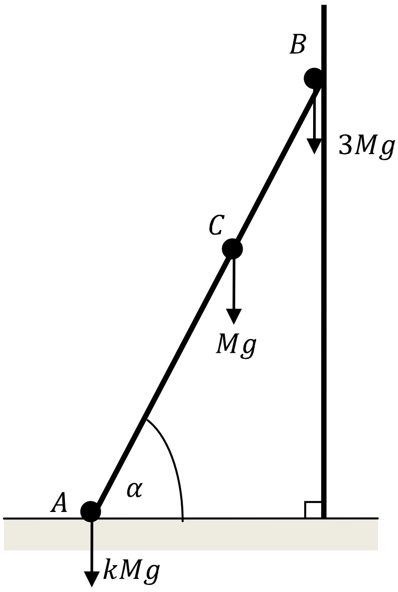
$$\Rightarrow (50 + 240) \times cos\ \alpha = (5 \times 45) \times sin\ \alpha$$

$$\Rightarrow \frac{(50 + 240)}{(5 \times 45)} = \frac{sin\ \alpha}{cos\ \alpha} = \tan \alpha$$

$$\Rightarrow \alpha = \tan^{-1} \frac{290}{225} = 52.2°$$

Example

A ladder AB of mass M and length $4a$ is placed against a smooth vertical wall and on a rough horizontal floor, such that the ladder is in a vertical perpendicular plane with respect to the wall. A particle of mass M is placed at point C on the ladder such that $BC = a$. Two further masses kM and $3M$ are attached to the ladder at A and B, respectively. The coefficient of friction between the ground and the ladder is $\frac{1}{4}$ and the ladder makes an angle α with the horizontal floor such that $\tan \alpha = \frac{17}{8}$. Given that these loads and the ladder are in equilibrium, find the possible range for the values of k.

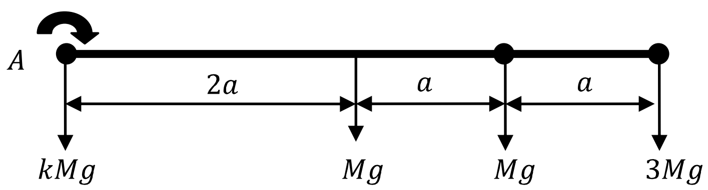


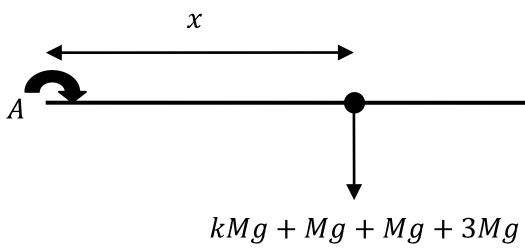
The ladder and loads, in this example, are designed to illustrate how more involved problems can be solved by reducing the system of masses to an effective mass acting through the centre of mass for the ladder together with the particles attached to the ladder.

While the ladder and masses are ultimately placed against a wall at an angle, the underlying collection of masses can be reduced to a single mass acting through the centre of mass. This principle is already applied to the concept of a uniform rod, where the mass representing the ladder acts through the geometric centre for the rod. We now wish to find the position on the ladder where a single mass attached to a light rod would produce the same result as analysing the masses and ladder as separate entities.

To calculate the centre of mass, the idea is to determine the turning effect about some point of the heavy rod (uniform rod-with-mass) and the three masses and determine the moment for a particle with mass equal to the total mass (the ladder plus the three masses) attached to a light rod, such that the moments for these two systems of masses and rods are exactly the same.

Ladder of length $4a$ of Mass M





The ladder and masses can be placed on the horizontal ground for the convenience of the calculation. A similar more involved calculation could be performed with the ladder against the wall, but the outcome for the centre of mass must be the same, which will be shown by solving the problem as stated by reducing the system of masses to a single effective mass and by direct solution treating each mass individually.

Placing the ladder and masses on the horizontal floor, the moment of a force is the magnitude of the force times the perpendicular distance to the line of action of the force. All weights act vertically, hence the desire to lay the ladder flat. Taking moments about the point A for each of these two equivalent mechanical systems yields:

For the heavy rod and masses:

$$moment\ about\ A = (kMg)(0) + (Mg)(2a) + (Mg)(3a) + (3Mg)(4a)$$

For the light rod and single mass:

$$moment\ about\ A = (kMg + Mg + Mg + 3Mg)(x)$$

If these two systems are equivalent the moments about A must be equal. Thus,

$$(kMg + Mg + Mg + 3Mg)(x) = (kMg)(0) + (Mg)(2a) + (Mg)(3a) + (3Mg)(4a)$$

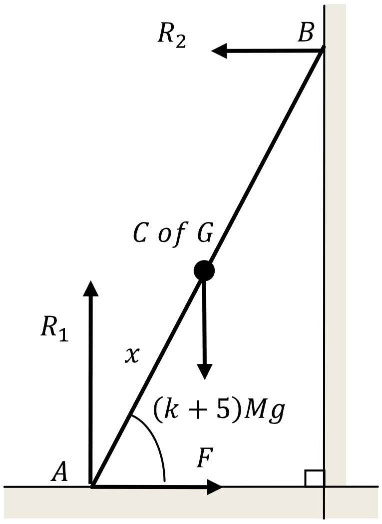
$$\Rightarrow x = \frac{(kMg)(0) + (Mg)(2a) + (Mg)(3a) + (3Mg)(4a)}{(kMg + Mg + Mg + 3Mg)}$$

$$\Rightarrow x = \frac{(kM)(0) + (M)(2a) + (M)(3a) + (3M)(4a)}{(kM + M + M + 3M)}$$

Those studying statistics will recognise the quotient as a weighted mean where the mass is equivalent to the frequency.

$$\Rightarrow x = \frac{17a}{(k+5)}$$

The original problem is then reduced to a light rod and a single particle:



The solution to the original problem then proceeds by listing the equations generated by resolving the forces vertically and horizontally as indicated on the diagram, the relationship between friction force and the normal reaction force of the ladder on the floor and taking moments about A.

Resolving vertically:

$$R_1 - (k+5)Mg = 0 \quad \text{... } (1)$$

Resolving horizontally:

$$F - R_2 = 0 \quad \text{... } (2)$$

Finally taking moments about A:

$$(x\cos\alpha)\,(k+5)Mg - (R_2)(4a)\sin\alpha = 0 \quad \text{... } (3)$$

Coefficient of friction:

$$F \le \mu R_1 \quad \text{... } (4)$$

Equation (1) yields $R_1 = (k+5)Mg$, therefore

$$F \le \frac{1}{4}(k+5)Mg \quad \text{... } (5)$$

Equation (2) shows that $F = R_2$ and substituting for x and F Equation (3) gives

$$(\frac{17a}{(k+5)}\cos\alpha)\,(k+5)Mg - F(4a)\sin\alpha = 0$$

$$\Rightarrow F = \frac{1}{4a}\frac{17a}{(k+5)}\frac{\cos\alpha}{\sin\alpha}(k+5)Mg$$

Since $\tan\alpha = \frac{17}{8}$,

$$\Rightarrow F = \left(\frac{1}{4}\right)\left(\frac{17}{1}\right)\left(\frac{8}{17}\right)Mg = 2Mg$$

Substituting $F = 2Mg$ into Equation (5) provides the inequality

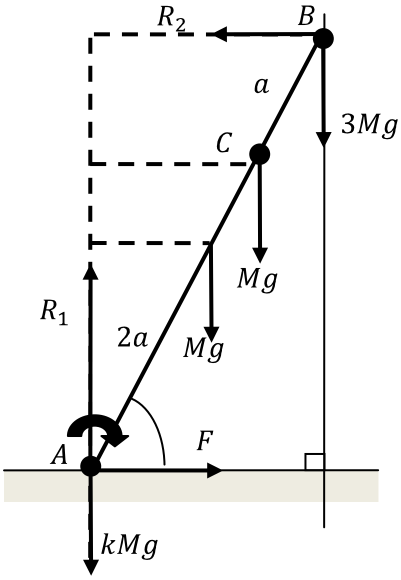
$$2Mg \leq \frac{1}{4}(k+5)Mg$$

$$\Rightarrow 8 \leq (k+5)$$

$$\therefore 3 \leq k$$

For equilibrium, the ladder must have a mass placed at A where, $k \geq 3$.

The alternative to reducing the ladder and masses to a single mass is the direct approach where all the forces for each explicit particle and the ladder are detailed on the diagram. The solution proceeds with exactly the same steps, the difference being the complexity of the equations generated from resolving vertically and taking moments about the point A. Taking a complex system and reducing the system to an equivalent simpler system is a common theme in higher mathematics, and thinking in terms of the centre of mass is a good example illustrating this approach. It also explains why calculating the centre of mass for complex rigid bodies features in most mechanics courses.



Resolving vertically:

$$R_1 - (kMg + Mg + Mg + 3Mg) = 0$$

Moments about A:

$$2a\cos\alpha\,(Mg) + 3a\cos\alpha\,(Mg) + 4a\cos\alpha\,(3Mg) - 4a\sin\alpha\,(R_2) = 0$$

These equations should be compared to Equations (1) and (3) above. The result obtained from these equations is the same as when the ladder and particles are reduced to a single mass acting at the centre of mass for the system.

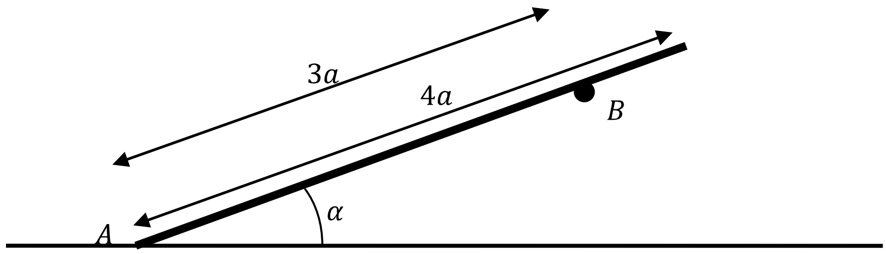
Example

A uniform rod of length $4a$ and mass M rests in equilibrium with one end in contact with a rough horizontal surface at A, and is supported by a smooth peg at position B where $AB = 3a$. When in limiting equilibrium, the rod makes an angle α with the horizontal surface. Show that

a) The normal reaction force at A is $Mg\frac{1}{3}(1 + 2\sin^2\alpha)$.

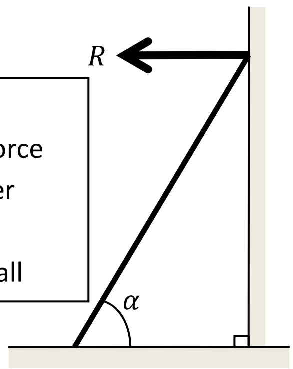
b) The coefficient of friction μ can be expressed in terms of the angle $\alpha$by

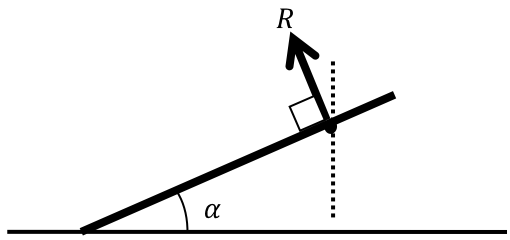
$$\mu = \frac{\sin 2\alpha}{1 + 2\sin^2\alpha}$$



The essential difference between a problem involving a ladder leaning *against* a wall and a rod *supported* by a peg is at the contact point for the rod. If the peg and the wall are smooth, that is the contact point between the rod and either the wall or the peg has no friction force therefore only involves a normal reaction force, the difference between these two problems is the direction for the reaction force. The normal reaction force of the ladder in contact with a vertical wall is horizontal, while for the smooth peg, the direction for the normal reaction force depends on the angle between the rod and the floor.

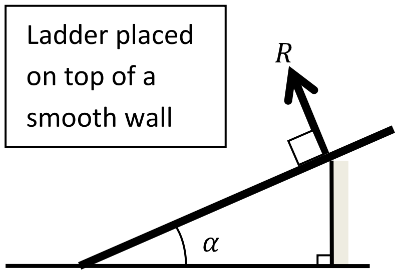
Normal reaction force for a ladder against a smooth wall





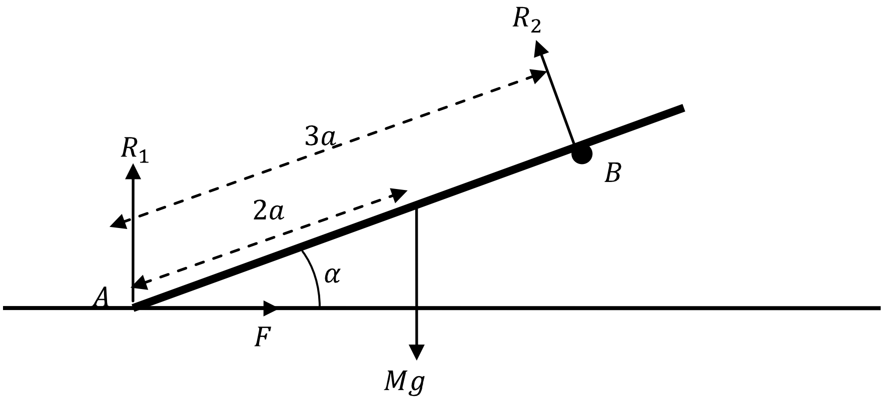
Normal reaction force for a rod supported by a smooth peg

If a ladder were placed on top of a smooth wall, the problem reduces to that of a smooth peg.



Solution

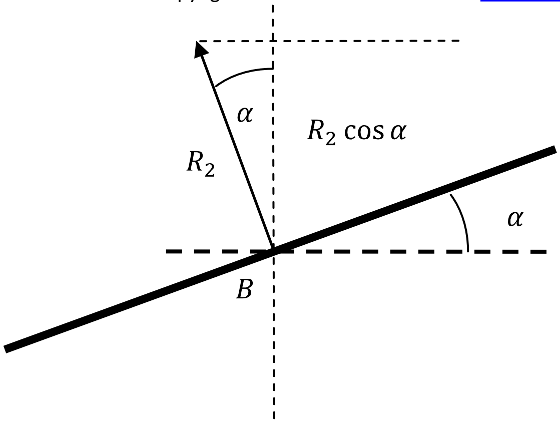
The most important part of the solution for the uniform rod supported by a smooth peg is to draw a diagram annotated with all the force vectors.



a) The normal reaction force at A is $Mg\frac{1}{3}(1 + 2\sin^2\alpha)$.

The first part to the question is intended to guide the solution in a direction leading to the answer for part b).

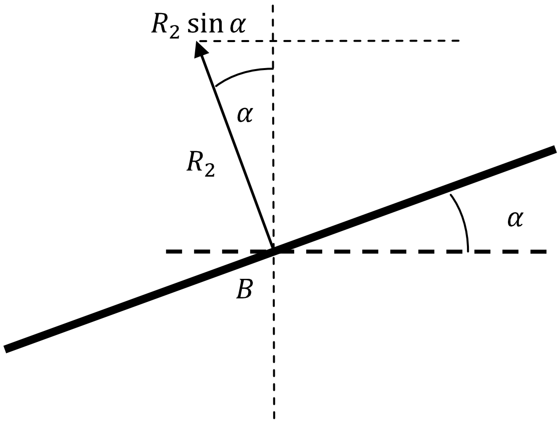
The pattern common to solving statics problems involves resolving the forces in two perpendicular directions and taking moments about an appropriate point to obtain equations relating the key information desired. Since the normal reaction force at A is the information requested first, resolving the force vectors in the vertical direction and, for equilibrium, equating to zero would be a good first step in the solution.



Resolving the forces vertically:

$$R_1 - Mg + R_2 \cos\alpha = 0$$

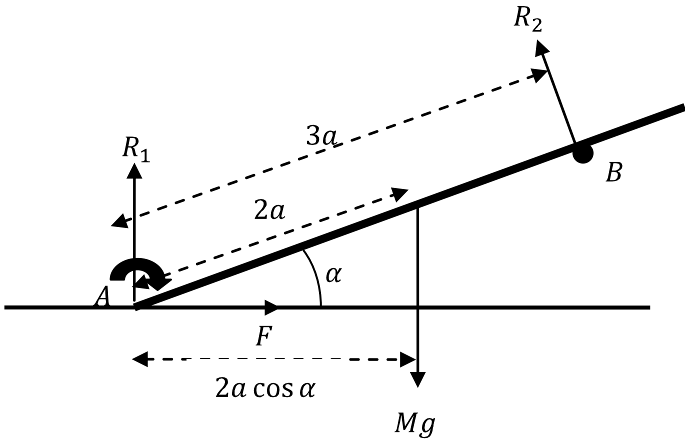
$$R_1 = Mg - R_2 \cos\alpha \quad \cdots (1)$$



Resolving the forces Horizontally:

$$F - R_2 \sin\alpha = 0$$

$$\Rightarrow F = R_2 \sin\alpha \quad \cdots (2)$$



Since F *and* R_1 act through A taking moments about A:

$$(Mg)2a\cos\alpha - (R_2)(3a) = 0$$

$$\Rightarrow (Mg)2a\cos\alpha = (R_2)(3a)$$

$$\Rightarrow R_2 = \frac{2}{3} Mg\cos\alpha \quad \cdots (3)$$

The expression for the normal reaction force at A is therefore obtained by substituting Equation (3) into Equation (1)

$$R_1 = Mg - (\frac{2}{3} Mg\cos\alpha)\cos\alpha$$

$$\Rightarrow R_1 = Mg\frac{1}{3}(3 - 2\cos^2\alpha)$$

And since $\cos^2\alpha = 1 - \sin^2\alpha$

$$\Rightarrow R_1 = Mg\frac{1}{3}(3 - 2(1 - \sin^2\alpha))$$

$$\Rightarrow R_1 = Mg\frac{1}{3}(1 + 2\sin^2\alpha) \quad \cdots (4)$$

b) The coefficient of friction μ can be expressed in terms of the angle α.

For limiting equilibrium

$$F = \mu R_1 \quad \cdots (5)$$

Substituting Equation (2) into Equation (5)

$$R_2 \sin\alpha = \mu R_1$$

Substituting from Equation (3), $R_2 = \frac{2}{3}Mg\cos\alpha$ and from Equation (4) $R_1 = Mg\frac{1}{3}(1 + 2\sin^2\alpha)$

$$\Rightarrow (\frac{2}{3}Mg\cos\alpha)\sin\alpha = \mu(Mg\frac{1}{3}(1 + 2\sin^2\alpha))$$

$$\Rightarrow (2\cos\alpha)\sin\alpha = \mu(1 + 2\sin^2\alpha)$$

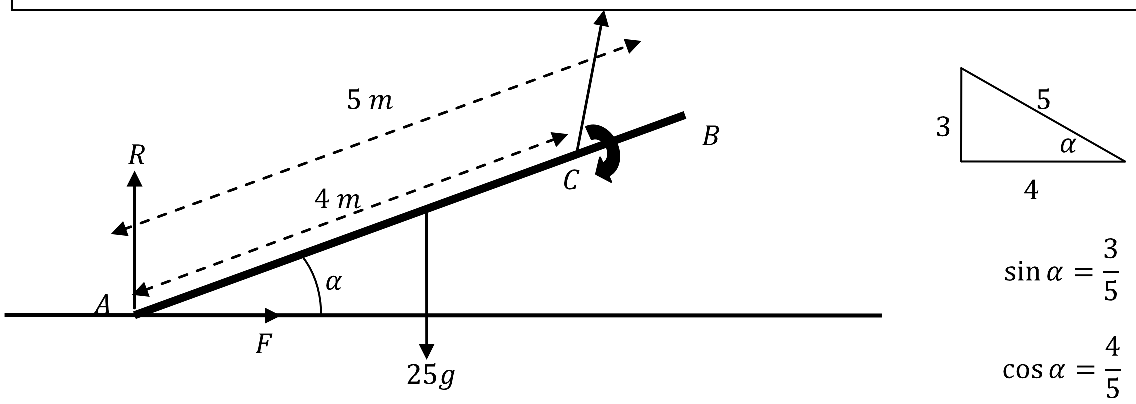
And since $\sin 2\alpha = 2\sin\alpha\cos\alpha$,

$$\Rightarrow \sin 2\alpha = \mu(1 + 2\sin^2\alpha)$$

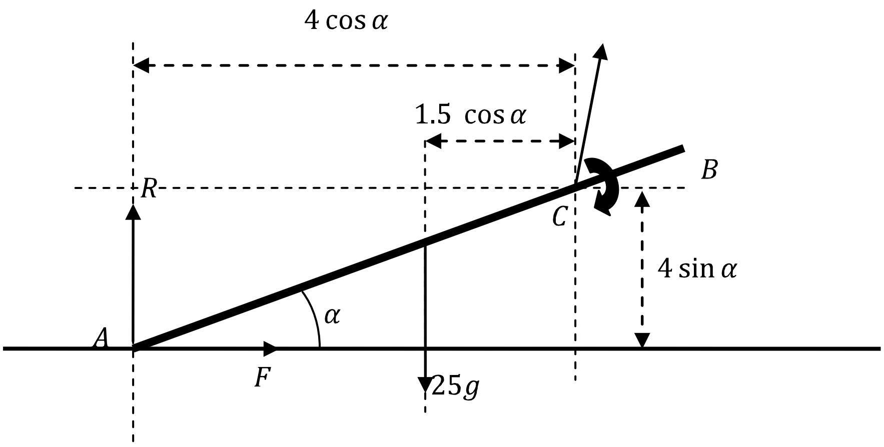
$$\Rightarrow \mu = \frac{\sin 2\alpha}{(1 + 2\sin^2\alpha)}$$

Example

A uniform rod AB of length $5\ m$ and mass $25\ kg$ rests with one end A on a rough horizontal plane. A force is applied to the rod at a point C which is $4\ m$ from A such that the rod is held in limiting equilibrium at an angle α to the horizontal, where $\sin\alpha = \frac{3}{5}$. The line of action of the force at C is in the same vertical plane as the rod. Given that the coefficient of friction between the rod and the horizontal plane is $\frac{7}{12}$, find the magnitude of the normal reaction force of the ground on the rod at A.



The force acting at C is not specified by a magnitude or direction. Since so little is known about the force at C the solution is likely to involve taking moments about C, as in so doing the resulting equation for equilibrium of the rod will not require any knowledge about the force at C. Further, since the coefficient of friction is given, the friction force is known in terms of the normal reaction at A, so taking moments about C and substituting $F = \mu R$ into the resulting equation allows the normal reaction to be determined.



For equilibrium Moments about C $\quad R\ 4\cos\alpha - 25g\ 1.5\ \cos\alpha - F\ 4\sin\alpha = 0$

Since $F = \frac{7}{12}R$, $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$,

$$R\ 4\left(\frac{4}{5}\right) - 25g\ 1.5\ \left(\frac{4}{5}\right) - \left(\frac{7}{12}\right)R\ 4\left(\frac{3}{5}\right) = 0$$

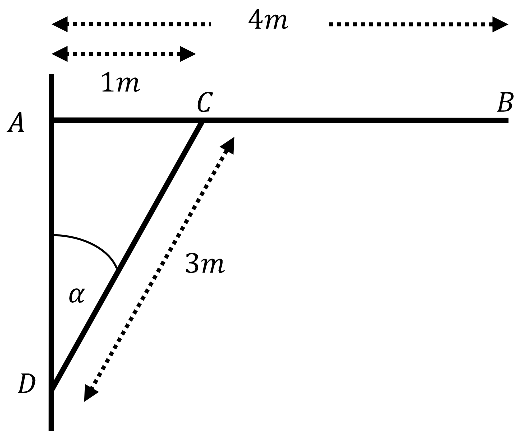
$$\Rightarrow R\ (16) - R\ (7) - g\ \ (150) = 0$$

$$\Rightarrow R\ = g\ \left(\frac{150}{9}\right) = 163\ N$$

Example

A uniform rod AB of length $4m$ and mass $20\ kg$ is smoothly hinged to a vertical wall at A. A light rod of length $3m$ is freely jointed to the rod AB at C and fixed to the wall vertically below A at D. The rod CD is positioned with $AC\ =\ 1m$ so that the rod AB is held horizontally in equilibrium.

a) Determine the thrust in the rod CD.
b) The magnitude of the force exerted by the wall on the rod at A.



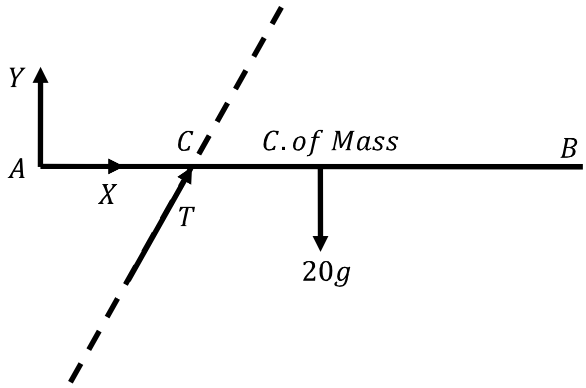
The problem as stated is aimed at maintaining the uniform rod AB in equilibrium in a horizontal position. To achieve this aim, all the forces acting on the rod AB must be introduced into a diagram for the rod AB.

The active force on the rod AB is the weight of the rod 20g acting through the centre of mass for the uniform rod. Since the rod is uniform, the centre of mass acts through the geometric centre of the rod and is therefore $2m$ from A.

Two passive forces, one at the smooth hinge at A and the second the thrust of the rod CD at C, must also be included as external forces acting on the rod AB. These passive forces act in response to the rod's weight, and are only present because of active forces such as the weight of the rod AB. If the rod were light, no force would exist at the hinge and no thrust would act in the rod CD.

A smooth hinge exerts a force on the rod AB from the wall with magnitude and direction as yet to be determined. To accommodate the reaction force at A, two perpendicular forces are introduced which represent the components of the reaction force $\boldsymbol{R}$ at A, or in vector notation, $\boldsymbol{R} = X\boldsymbol{i} + Y\boldsymbol{j}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the horizontal and vertical directions.

Thrust exerted by the rod is a passive force acting against the force attempting to compress the rod. The thrust T acts in a direction parallel to the rod CD and through the point C on the rod AB.

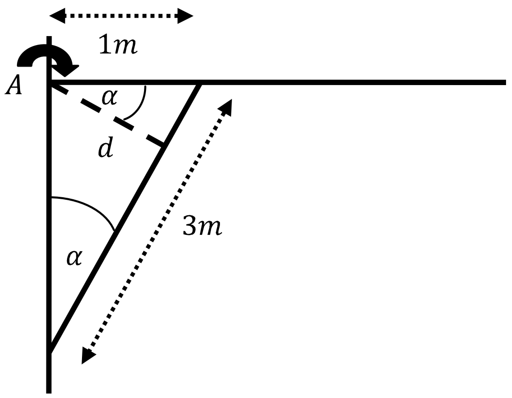


Solution

a) Determine the thrust in the rod CD.

Since the reaction force in the hinge acts through A, taking moments about A and, for equilibrium, equating to zero provides an equation involving the weight of the rod AB and the thrust only.

The perpendicular distance between the line of action of the weight and the point A is $2m$. The perpendicular distance d between the line of action for the thrust and A must be calculated from the geometry of the rods and vertical wall.



$$\sin\alpha = \frac{1}{3}$$

$$\cos\alpha = \frac{\sqrt{3^2 - 1^2}}{3} = \frac{2\sqrt{2}}{3}$$

$$\therefore \frac{d}{1} = \cos\alpha = \frac{2\sqrt{2}}{3}$$

Moments about A:

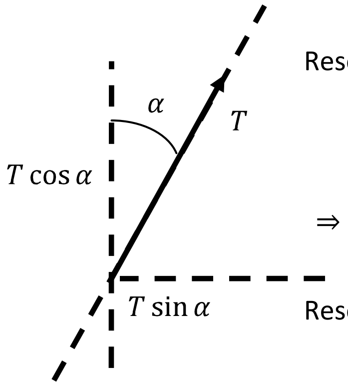
$$20g(2) - T(d) = 0$$

$$\Rightarrow 20g(2) = T\left(\frac{2\sqrt{2}}{3}\right)$$

$$\Rightarrow T = \frac{60g}{\sqrt{2}} = 415.8\ N$$

b) The magnitude of the force exerted by the wall on the rod at A.

The reaction force at the smooth hinge is defined in terms of two perpendicular components X and Y, therefore the magnitude of the reaction force $\boldsymbol{R} = X\boldsymbol{i} + Y\boldsymbol{j}$, is $|\boldsymbol{R}| = \sqrt{X^2 + Y^2}$. These component forces are obtained by resolving the forces on the rod AB in the horizontal and vertical directions.



Resolving vertically,

$$Y + T\cos\alpha - 20g = 0$$

$$\Rightarrow Y = 20g - T\cos\alpha = 196 - 415.8\left(\frac{2\sqrt{2}}{3}\right) = -196\ N$$

Resolving horizontally,

$$X + T\sin\alpha = 0$$

$$X = -T\sin\alpha = -415.8\left(\frac{1}{3}\right) = -138.6\ N$$

The negative sign for the components to the reaction force of the hinge indicates the forces as marked on the diagram are actually in the opposite direction.

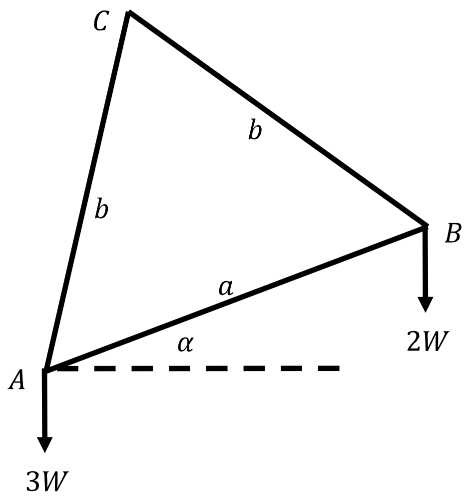
The magnitude for the reaction force of the hinge at A is therefore,

$$|\boldsymbol{R}| = \sqrt{X^2 + Y^2} = \sqrt{(-138.6)^2 + (-196)^2} = 240\ N$$

Example

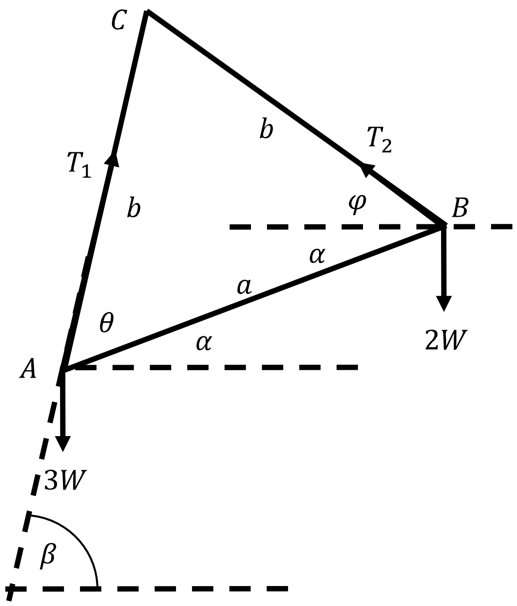
A particle of weight $3W$ is attached to the end A of a light rod AB of length a. A second particle of weight $2W$ is attached to the opposite end B of the rod. Two light inextensible strings each of length b are attached to the ends of the rod before being fixed to a point C, from which the rod is hung. Show that if the rod is in equilibrium, the angle between the rod and the horizontal is α where

$$\tan\alpha = \frac{a}{5\sqrt{4b^2 - a^2}}$$



The active forces are due to particles located at A and B on the light rod. When in equilibrium, these active forces must be balanced by passive forces in the two light inextensible strings. The strings are fixed at C and attached at each end of the rod, therefore each string will cause a force to act on the rod due to the strings resistance to being stretched. A force resisting elongation in a string is referred to as tension and is illustrated using an arrow pointing into the string.

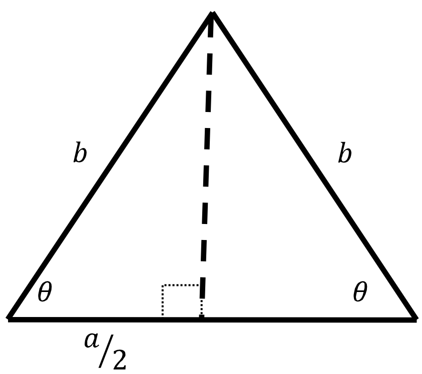
The solution to the problem of hanging a rod using two strings, as usual, involves drawing a diagram to illustrate the forces acting on the rod and particles, however, the importance of an accurate well drawn diagram is clearly shown by this example. Without a clear geometric understanding for the system of strings, particles and rod, the solution is more difficult to see. The problem is posed to extract a higher than usual mathematical content for mechanics questions, but would certainly require an even higher mathematical intuition without a pictorial understanding of the components involved.

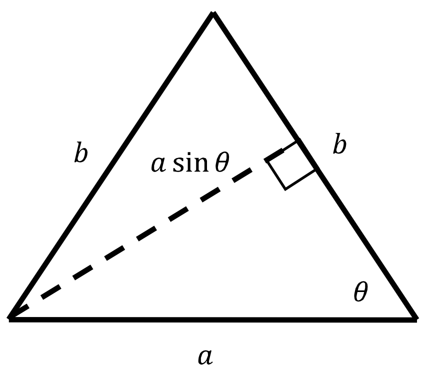


Geometric points:

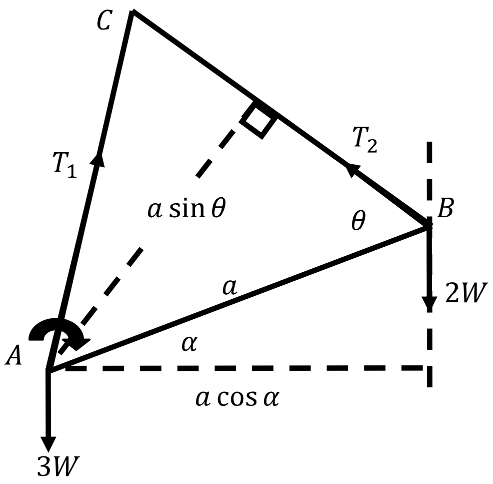
1. The triangle ABC is isosceles, with two side of equal length and therefore the angles $\angle CAB$ and $\angle ABC$ are equal angles θ, say.
2. The angle between T_2 and the horizontal $\varphi = \theta - \alpha$.
3. The angle between T_1 and the horizontal $\beta = \theta + \alpha$.

The solution involves observing the structure within the mechanical system; specifically the four forces are divided between two points of action, therefore taking moments about each of the two points A and B yields two equations for the tensions T_1 and T_2 in terms of the angle α. The angle θ is clearly determined by the isosceles triangle, as follows.





Using the geometry for the isosceles triangle: $\cos\theta = \frac{a/2}{b}$, $\sin\theta = \frac{\sqrt{b^2 - \left(\frac{a}{2}\right)^2}}{b}$ and $\tan\theta = \frac{\sqrt{b^2 - \left(\frac{a}{2}\right)^2}}{a/2}$.



Taking moments about A and, for equilibrium, equating to zero:

$$-T_2 a\sin\theta + 2Wa\cos\alpha = 0 \quad \cdots(1)$$

Similarly, taking moments about B and, for equilibrium, equating to zero:

$$T_1 a\sin\theta - 3Wa\cos\alpha = 0 \quad \cdots(2)$$

These two equations involve three unknowns, T_1, T_2 and α, therefore a third equation relating these values is required.

Resolving horizontally and, again, using the condition for equilibrium that the resultant force in any direction must be zero,

$$T_1\cos\beta - T_2\cos\varphi = 0$$

Since $\beta = \theta + \alpha$ and $\varphi = \theta - \alpha$,

$$T_1\cos(\theta+\alpha) - T_2\cos(\theta-\alpha) = 0$$

Using the trigonometric identities

$$\cos(\theta+\alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

And

$$\cos(\theta-\alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$T_1(\cos\theta\cos\alpha - \sin\theta\sin\alpha) - T_2(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 0 \quad \cdots(3)$$

Equations (1) and (2) yield

$$\frac{T_2}{2}\sin\theta = W\cos\alpha$$

$$\frac{T_1}{3}\sin\theta = W\cos\alpha$$

$$\therefore \quad T_2 = \frac{2}{3}T_1$$

Substituting for T_2 into Equation (3),

$$T_1(\cos\theta\cos\alpha - \sin\theta\sin\alpha) - \frac{2}{3}T_1(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 0$$

$$\Rightarrow 3\,(\cos\theta\cos\alpha - \sin\theta\sin\alpha) - 2(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 0$$

$$\Rightarrow (3\cos\theta\cos\alpha - 2\cos\theta\cos\alpha) - (3\sin\theta\sin\alpha + 2\sin\theta\sin\alpha) = 0$$

$$\Rightarrow \cos\theta\cos\alpha - 5\sin\theta\sin\alpha = 0$$

$$\Rightarrow \frac{\cos\theta}{5\sin\theta} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

Since, $\tan\theta = \frac{\sqrt{b^2 - \left(\frac{a}{2}\right)^2}}{a/2}$

$$\Rightarrow \tan\alpha = \frac{1}{5}\frac{1}{\tan\theta} = \frac{1}{5}\frac{a/2}{\sqrt{b^2 - \left(\frac{a}{2}\right)^2}} = \frac{a}{5\sqrt{4b^2 - a^2}}$$